Application of Dispersion Relations to Pion-Nucleon Scattering*

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The generalized Kramers-Kronig dispersion relations for charged bosons are used to treat the problem of pion-nucleon scattering. The complications associated with the charge of the pions are discussed. The importance of a "bound state" corresponding to the neutron is emphasized and its constrbution to the scattering amplitude is computed rigorously, assuming only that pions are pseudoscalar and that the interaction with nucleons is charge-independent. The connection between our exact dispersion relations and the approximate equations for pion-nucleon scattering given by Low is discussed. A rigorous effective-range relation is derived.

1. INTRODUCTION

In the preceding paper,[1] a new derivation of dispersion relations of the Kramers-Kronig type has been given which is of sufficient generality that the problem of the scattering of charged pions by nucleons may be treated. In the present paper we wish to specialize the results of I to this case and cast them into a form suitable for comparison with experiment. The application of dispersion relations to pion-nucleon scattering was first suggested by R. Karplus and M. Ruderman (preprint, January, 1955). Their results, however, could not be applied to the description of the scattering of charged pions by protons. In view of the importance of this tool in the analysis of experimental data, it was felt worth while to discuss the problem in detail.

In Sec. 2, we shall give the explicit formulas appropriate for the description of the scattering of positive and negative pions by protons. There appears in these dispersion relations a term corresponding to a rather unusual bound state, the neutron, whose contribution may be expressed in terms of the strength of the pion-nucleon interaction. In Sec. 3, the results will be discussed and the connection between our work and that of Low[2] will be developed.

2. DISPERSION RELATIONS FOR PION-NUCLEON SCATTERING

The general dispersion relations given in I must now be written explicitly for the case of pion-nucleon scattering. Regarded as a matrix in nucleon isotopic spin space, the forward-scattering amplitude describing the scattering of a meson with isotopic spin index β into one with isotopic spin index α ($\alpha,\beta=1, 2, 3$) may be written as

$$T_{\alpha\beta}(\omega)=\delta_{\alpha\beta}T^{(1)}(\omega)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]T^{(2)}(\omega), \qquad (2.1)$$

where ω is the total meson energy in the laboratory system and we have assumed charge independence. Both $T^{(1)}$ and $T^{(2)}$ have a dispersive and an absorptive part:

$$T^{(j)}(\omega)=D^{(j)}(\omega)+i\epsilon(\omega)A^{(j)}(\omega),\quad j=1,2. \qquad (2.2)$$

We have introduced the factor $\epsilon(\omega)$ in order that our notation coincide with that of I. We consider here only positive energies, so it is of no importance. In place of the representation we have been using it is convenient to introduce the amplitudes for pure isotopic spin states, namely

$$\begin{aligned}T^{\frac{3}{2}}(\omega)&=T^{(1)}(\omega)-T^{(2)}(\omega),\\ T^{\frac{1}{2}}(\omega)&=T^{(1)}(\omega)+2T^{(2)}(\omega).\end{aligned} \qquad (2.3)$$

We have assumed that the nucleon charge state does not change and we shall assume that it is a proton. It follows immediately from Eq. (2.3) that

$$\begin{aligned}T^{(1)}(\omega)&=\tfrac{1}{2}[T_-(\omega)+T_+(\omega)],\\ T^{(2)}(\omega)&=\tfrac{1}{2}[T_-(\omega)-T_+(\omega)],\end{aligned} \qquad (2.4)$$

where $T_\pm(\omega)$ are the forward amplitudes for the coherent scattering of $\pi^\pm$ mesons by protons, i.e., $\pi^+\to\pi^+$, $\pi^-\to\pi^-$. We note that the four quantities $D^{(j)}$ and $A^{(j)}$ are real since the nucleon charge is not changed. [See discussion following I-(2.22).] Thus $D^{(j)}$ and $A^{(j)}$ are respectively the real and imaginary parts of the forward scattering amplitudes defined by Eq. (2.4). Since we have our simple dispersion relations [I-(2.40), I-(2.41)] only for the quantities $D^{(j)}$ and $A^{(j)}$, it is evident that neither the pure isotopic spin amplitudes nor the charged meson amplitudes satisfy them separately. The mathematical reason is, of course, that these more physical amplitudes have no simple behavior of evenness or oddness when $\omega\to-\omega$. We shall return to this point later in the discussion.

We now write the dispersion relations for $T^{(1)}$ and $T^{(2)}$ in the following form:

$$\tfrac{1}{2}[D_-(\omega)+D_+(\omega)]-\tfrac{1}{2}[D_-(\mu)+D_+(\mu)] =\frac{2(\omega^2-\mu^2)}{\pi}\int_0^\infty d\omega'\frac{\omega'\frac{1}{2}[A_-(\omega')+A_+(\omega')]}{(\omega'^2-\mu^2)(\omega'^2-\omega^2)}, \qquad (2.5)$$

$$\tfrac{1}{2}[D_-(\omega)-D_+(\omega)]-\frac{\omega}{\mu}\tfrac{1}{2}[D_-(\mu)-D_+(\mu)] =\frac{2\omega(\omega^2-\mu^2)}{\pi}\int_0^\infty d\omega'\frac{\frac{1}{2}[A_-(\omega)-A_+(\omega')]}{(\omega'^2-\mu^2)(\omega'^2-\omega^2)}. \qquad (2.6)$$

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[1] M. L. Goldberger, preceding paper [Phys. Rev. **99**, 979 (1955)]; hereafter referred to as I.

[2] F. E. Low, Phys. Rev. **97**, 1392 (1955).

These are just Eqs. I-(2.32) and I-(2.33), with $\omega_0=\mu$. We have introduced the obvious notation $T_\pm(\omega)=D_\pm(\omega)+i\epsilon(\omega)A_\pm(\omega)$. Recall that the integrations over the singularities are to be carried out in the sense of Cauchy principal values. All quantities appearing in Eqs. (2.5) and (2.6) are to be computed in the laboratory system in which the proton is initially at rest.

For the region of integration extending from μ to ∞, we have the relations

$$\sigma_\pm(\omega)=(4\pi/k)A_\pm(\omega), \tag{2.7}$$

where $k=(\omega^2-\mu^2)^{\frac{1}{2}}$ and $\sigma_\pm(\omega)$ are the total cross sections for all processes originating from positive or negative pions incident upon a proton. In the integration region $0\leq\omega<\mu$, there is a contribution from a "bound state" corresponding to an intermediate state consisting of a neutron. This may be computed almost exactly from meson theory.

To find the bound state contribution we write out the amplitudes $A_\pm(\omega)$ as sums over a complete set of states, following I-(2.23), modified of course approximately to describe our charged-meson amplitudes:

$$A_\pm(\omega)=\pi\sum_{n,\mathbf{p}_n=\mathbf{k}}\{|\langle n|j_\pm(0)|p\rangle|^2\delta(E_p-E_n+\omega)$$
$$-|\langle n|j_\mp(0)|p\rangle|^2\delta(E_p-E_n-\omega)\}, \tag{2.8}$$

where $j_\pm(x)$ are the "currents" associated with the charged meson fields, defined by

$$\begin{aligned}(\mu^2-\Box^2)\phi(x)&=j_-(x),\\(\mu^2-\Box^2)\phi^*(x)&=j_+(x),\end{aligned} \tag{2.9}$$

and the connection between ϕ, ϕ^*, and the ϕ_α of I is

$$\begin{aligned}\phi&=(\phi_1-i\phi_2)/\sqrt{2},\\\phi^*&=(\phi_1+i\phi_2)/\sqrt{2}.\end{aligned} \tag{2.10}$$

It is evident that the only states $|n\rangle$ which can possibly contribute to an integral over ω in the region $0\leq\omega<\mu$ are those corresponding to a single nucleon state. For all others, even though we allow ω to be less than μ (or $k^2=\omega^2-\mu^2<0$) and consequently $E_n=(M_n{}^2+k^2)^{\frac{1}{2}}<M_n$, where M_n is the total rest mass of the states $|n\rangle$, the δ functions can never vanish. For the positive values of ω to which we have restricted ourself, only the second term in (2.8) can contribute and in particular, if $|p\rangle$ represents a proton state, by charge conservation only $\langle n|j_-(0)|p\rangle$ is different from zero, corresponding to the emission of a π^+ meson by a proton leading to a neutron state. The "energy" of this state is $E_n=M-\mu^2/2M$, corresponding to $\omega=\mu^2/2M$ and $k^2=-\mu^2+(\mu^2/2M)^2$. Consequently only $A_+(\omega)$ exhibits the δ function singularity for $0<\omega<\mu$.

Since the state $|n\rangle$ represents a single nucleon state the required matrix element in (2.8) may be expressed in terms of the vertex operator Γ_5 corresponding to the emission of a virtual positive meson by a proton leading to a neutron. According to Low,[3] we have to within terms of order $(\mu/M)^2$ the result

$$\langle n|j_-(0)|p\rangle=-ig\sqrt{2}\boldsymbol{\sigma}\cdot\mathbf{k}/2M, \tag{2.11}$$

and thus

$$A_+(\omega)=-2\pi g^2\frac{k^2}{(2M)^2}\delta\left(\omega-\frac{\mu^2}{2M}\right),\quad 0<\omega<\mu. \tag{2.12}$$

In these equations, g is the renormalized coupling constant of the symmetrical pseudoscalar theory (in Gaussian units, with $\hbar=c=1$) and M is the nucleon mass. Note that we always maintain the relation $k^2=\omega^2-\mu^2$, even when $\omega<\mu$. In other words, our scattering amplitudes are to be expressed entirely in terms of ω and then continued from $\omega>\mu$ to $\omega<\mu$. We observe that for $\omega>\mu$, only the first term in Eq. (2.8) contributes to $A_+(\omega)$, and it is a positive contribution, and as a result of the fact that $k^2=\omega^2-\mu^2=+(\mu^2/2M)^2-\mu^2$, $A_+(\omega)$ is also positive for $\omega<\mu$. The contribution of this neutron state to the real part of the forward amplitudes is numerically very important.

We now substitute Eqs. (2.7) and (2.12) into Eqs. (2.5) and (2.6) and also change the variables from energy to wave number. We find then, using the same letters to designate functions of k that had been used for functions of ω, the results

$$\tfrac{1}{2}[D_-(k)+D_+(k)]-\tfrac{1}{2}[D_-(0)+D_+(0)]=\frac{k^2}{4\pi^2}\int_0^\infty dk'$$
$$\times\frac{\sigma_-(k')+\sigma_+(k')}{k'^2-k^2}+2f^2\frac{k^2}{\omega^2-(\mu^2/2M)^2}\frac{1}{2M}, \tag{2.13}$$

$$\tfrac{1}{2}[D_-(k)-D_+(k)]-\frac{\omega}{\mu}\tfrac{1}{2}[D_-(0)-D_+(0)]=\frac{k^2\omega}{4\pi^2}\int_0^\infty\frac{dk'}{\omega'}$$
$$\times\frac{\sigma_-(k')-\sigma_+(k')}{k'^2-k^2}-2f^2\frac{k^2}{\omega^2-(\mu^2/2M)^2}\frac{\omega}{\mu}\frac{1}{\mu}. \tag{2.14}$$

We have introduced the small coupling constant $f=\mu g/2M$ characteristic of the pseudovector interaction. The energy dependence of the bound state term is exact, however the coefficient, $2f^2$, as we have remarked before, is accurate only to terms of order $(\mu/M)^2$. The final form of the dispersion relations for the individual amplitudes $D_\pm(k)$ is obtained by combining Eqs. (2.13) and (2.14):

$$D_+(k)-\frac{1}{2}\left(1+\frac{\omega}{\mu}\right)D_+(0)-\frac{1}{2}\left(1-\frac{\omega}{\mu}\right)D_-(0)$$
$$=\frac{k^2}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}\left[\frac{\sigma_+(\omega')}{\omega'-\omega}+\frac{\sigma_-(\omega')}{\omega'+\omega}\right]+\frac{2f^2}{\mu^2}\frac{k^2}{\omega-\mu^2/2M}, \tag{2.15}$$

$$D_-(k)-\frac{1}{2}\left(1+\frac{\omega}{\mu}\right)D_-(0)-\frac{1}{2}\left(1-\frac{\omega}{\mu}\right)D_+(0)$$
$$=\frac{k^2}{4\pi^2}\int_\mu^\infty\frac{d\omega'}{k'}\left[\frac{\sigma_-(\omega')}{\omega'-\omega}+\frac{\sigma_+(\omega')}{\omega'+\omega}\right]-\frac{2f^2}{\mu^2}\frac{k^2}{\omega+\mu^2/2M}. \tag{2.16}$$

[3] See reference 2, Eq. (3.8). It is worth noting that the form of this matrix element is essentially independent of any details of the pion-nucleon interaction, although we have used the terminology of the usual γ_5 theory, and depends only on parity and angular momentum considerations. It affords a definition of the coupling constant which measures the strength of the interaction.

We have reverted to the original variable, ω', in the integrals, although the wave number is more convenient for numerical evaluation.

3. DISCUSSION

The final forms of the dispersion relations, Eqs. (2.15) and (2.16), for the real part of the scattering amplitudes, $D_{\pm}(k)$, are quite different from what one obtains by naively applying a dispersion relation such as Eq. (2.15) to the individual amplitudes for the scattering of positive and negative pions. We should perhaps note that this is the correct procedure for the case of π^0 scattering. The underlying physical reason for the profound difference between the scattering of neutral and charged pions seems to be that in making the continuation of the scattering amplitudes to negative frequencies, the charge of the beam must be changed. Thus only the symmetric and antisymmetric combinations of amplitudes which we have considered have simple properties when we continue them to negative frequencies. It may in fact be shown that if we define two quantities $M_{\pm}(\omega)$ by the relations

$$M_{\pm}(\omega)\equiv D_{\pm}(\omega)+iA_{\pm}(\omega), \tag{3.1}$$

which coincide with $T_{\pm}(\omega)$ for $\omega>0$, that

$$M_{\pm}(\omega)=M_{\mp}^{*}(-\omega). \tag{3.2}$$

As was shown in I, it is the M's for which the dispersion relations, strictly speaking, hold, and it is this fact that leads to our fundamental relations Eqs. (2.5) and (2.6). In the case of neutral pions, because the π^0 is its own charge conjugate we have

$$M_0(\omega)=M_0^{*}(-\omega), \tag{3.3}$$

and consequently the appropriate dispersion relation is

$$D_0(\omega)-D_0(\mu)=\frac{2(\omega^2-\mu^2)}{\pi}\int_0^{\infty}d\omega'\omega'\frac{A_0(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\mu^2)}. \tag{3.4}$$

Our expressions for the amplitudes $D_{\pm}(k)$, Eqs. (2.15) and (2.16) bear a striking resemblance to the equations for pion-nucleon scattering recently proposed by Low. In fact, if one assumes that scattering occurs only in p-states and further considers the limit $M\rightarrow\infty$, and replaces our exact total cross sections by the total *elastic* cross sections, one gets two combinations of Low's three integral equations for the phase shifts.[2] We can obtain only those which do not involve spin flips, because there is no spin flip in the forward direction.[4] Our expressions are of course much more general although they are not equations for the determination of the complete scattering amplitudes.

We may also provide the exact form of the effective range relations which have been suggested by Chew and Low[5] by expanding the integrals in Eqs. (2.15) and (2.16) in powers of ω. This must be done with some care because of the singular nature of principal value integrals; however, the leading term is trivially obtained simply by setting ω equal to zero. Calling the left-hand sides of Eqs. (2.15) and (2.16) $L_{\pm}(k)$, we have the rigorous effective range relations

$$\left(\omega\mp\frac{\mu^2}{2M}\right)\frac{L_{\pm}(k^2)}{k^2}=\pm\frac{2f^2}{\mu^2}+\frac{(\omega\mp\mu^2/2M)}{4\pi^2}\int_{\mu}^{\infty}\frac{d\omega'}{k'}\frac{\sigma_{+}(\omega')+\sigma_{-}(\omega')}{\omega'}+\cdots. \tag{3.5}$$

Note that the effective range, r_e, defined by

$$r_e=\frac{\mu^2}{4\pi}\int_{\mu}^{\infty}\frac{d\omega'}{k'}\frac{\sigma_{+}(\omega')+\sigma_{-}(\omega')}{\omega'}, \tag{3.6}$$

is a positive quantity.

Note added in proof.—An interesting sum rule may be obtained by dividing Eq. (2.14) by ω and passing to the limit of $\omega\rightarrow\infty$. Within the framework of the assumptions already made in deriving the dispersion relations it can be shown that the difference $[D_{-}(\omega)-D(\omega)]/\omega$ either approaches a constant or goes to zero as $1/\omega^2$. Calling the quantity $\Delta(\omega)$, we have the result

$$\Delta(\mu)-\Delta(\infty)=\frac{4f^2}{\mu^2}+\frac{1}{2\pi^2}\int_{\mu}^{\infty}\frac{d\omega}{k}[\sigma_{-}(\omega)-\sigma_{+}(\omega)]. \tag{3.7}$$

Using the value of f^2 given by reference 5, namely 0.081, and the experimental data on the total cross sections one finds $\Delta(\mu)-\Delta(\infty)=0.18/\mu^2$ which is precisely the value given by Orear[6] for $\Delta(\mu)=2[a_1-a_3]/3\mu$. Thus to within the present experimental accuracy, we may conclude that $\Delta(\infty)=0$. We have not as yet been able to prove theoretically that $\Delta(\infty)$ must be zero, but it would not be unreasonable. If true, Eq. (3.7) then provides us with a sum rule relating the difference of the s-wave scattering lengths to the coupling constant and the total cross sections.

The results obtained in this paper have been applied to the phase-shift analysis of pion-nucleon scattering data by H. L. Anderson, W. Davidon, and U. Kruse. This work will be published shortly.

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[4] *Note added in proof.*—It has been shown by one of us (R. O.) and independently by W. Thirring (private communication) that the spin flip equations may also be deduced from the causality considerations of the preceding paper by studying the derivative of the scattering amplitude with respect to angle, evaluated in the forward direction.

[5] G. F. Chew and F. E. Low, Fifth Annual Rochester Conference on High-Energy Nuclear Physics, 1955 (to be published).

[6] J. Orear, Phys. Rev. **96**, 176 (1954).